PERFORMANCE ANALYSIS OF BANDED EQUALIZERS FOR OFDM SYSTEMS IN TIME-VARYING CHANNELS

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ABSTRACT

OFDM systems affected by severe time-varying channels greatly benefit from equalization schemes based on intercarrier interference (ICI) mitigation, which guarantee improved performance with respect to conventional one-tap equalizers. By exploiting a semi-analytical approach, this paper assesses the BER performance of the so-called banded equalizers, i.e., those equalizers that take into account only the ICI produced by the closest subcarriers. Specifically, by exploiting the Gaussian approximation of the residual ICI at the equalizer output, we evaluate the BER of block linear equalizers designed under a minimum mean-squared error (MMSE) criterion. Simulation results show a very good agreement with the theoretical analysis. In addition, we derive a lower bound and an approximate upper bound for the BER of block decision-feedback equalizers.

Index Terms— Performance analysis, OFDM, time-varying channels

1. INTRODUCTION

A key feature of multicarrier systems with cyclic prefix (CP), such as OFDM, is the capability of converting a time-invariant (TI) frequency-selective channel in a set of parallel orthogonal frequency-flat channels. However, in time-varying (TV) channels, such as those experienced in high-mobility communications, the orthogonality among the OFDM subcarriers is destroyed, due to the intercarrier interference (ICI) generated by the channel Doppler spread [1]-[3]. If left uncompensated, the ICI highly degrades the BER performance of OFDM, as testified by [4] and [5], which analyze the BER of the conventional one-tap OFDM equalizer.

To mitigate the ICI effects, different equalization schemes have been proposed (see [6]-[8], and the references therein). These schemes, though more complex than one-tap equalization, are quite effective in reducing the BER floor caused by the ICI, and therefore are suitable for high-mobility communications. However, for these equalizers, a theoretical BER characterization is still lacking, and the BER in [6]-[8] is illustrated by means of simulation results, which are usually time-consuming and valid only for some specific scenarios.

Thus, in this paper, we focus on the analytical BER performance of the so-called banded equalizers presented in [8]. The specific feature of the banded equalizers is that they try to combat only the ICI that comes from adjacent subcarriers. This is motivated by the fact that most of the ICI is generated by few closest subcarriers [7]. Since the ICI produced by the faraway subcarriers is ignored, the frequency-domain channel matrix becomes banded. Relying on this matrix structure, many equalization algorithms can be exploited to reduce complexity [7][8]. Specifically, the algorithms in [8], which exploit a band LDL decomposition, have a computational complexity that is only linear in the number of subcarriers, differently from other more complex alternatives [6][7].

In order to derive the BER performance of the MMSE linear equalizer (LE) presented in [8], we adopt a semi-analytical approach. First, we make use of the Gaussian approximation of the ICI, already exploited in [4], to express the BER conditioned on a given channel realization. Second, we average the obtained results over a certain number of typical computer-generated channel realizations. This procedure is the main difference with the fully-analytical methods, where the average is performed by mathematical integration over the channel statistics. Although closed-form analytical BER expressions are generally attractive, they exist or are possible to derive for specific constellations and channel statistics (e.g., Rayleigh, Rice, etc.). On the contrary, one of the merits of semi-analytical approaches is their generality, because any constellation format and channel statistics can be assumed. On the other hand, semi-analytical methods are by far faster than the complete system simulation.

We also consider the BER performance of banded decision-feedback equalizers (DFE) [8], which are known to suffer from error propagation. Since the number of error events can be quite large, for DFE even an approximate BER analysis is prohibitive [9]. In this paper, we derive both a lower bound and an approximate upper bound. The lower bound is obtained by disregarding the presence of error propagation, while the approximate upper bound is obtained by following the approach of [10]. To this end, we still use the semi-analytical method previously described. Simulation results show that both the bounds correctly follow the exact BER performance.

2. BANDED EQUALIZATION FOR OFDM

We consider an OFDM system with N subcarriers. We assume time and frequency synchronization, and a CP length L greater than the maximum delay spread of the channel. We assume that the receiver can apply an $N\times 1$ time-domain window $\mathbf{w}$, after the CP removal and before the FFT, in order to reduce the Doppler spreading. At the receiver, after the FFT, the generic received OFDM block can be expressed by [8]

$$\underline{\mathbf{z}}_{\mathrm{W}} = \underline{\mathbf{\Lambda}}_{\mathrm{W}}\underline{\mathbf{a}} + \underline{\mathbf{n}}_{\mathrm{W}} = \underline{\mathbf{C}}_{\mathrm{W}}\underline{\mathbf{\Lambda}}\underline{\mathbf{a}} + \underline{\mathbf{n}}_{\mathrm{W}}\ , \qquad (1)$$

where $\underline{\mathbf{z}}_{\mathrm{W}}$ is the $N\times 1$ frequency-domain received vector, $\underline{\mathbf{\Lambda}}_{\mathrm{W}} = \mathbf{F}\mathbf{\Delta}_{\mathrm{W}}\underline{\mathbf{H}}\mathbf{F}^{H}$ is the $N\times N$ frequency-domain channel matrix, which includes the presence of windowing by the $N\times N$ diagonal

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matrix $\boldsymbol{\Delta}_{\mathrm{W}} = \mathrm{diag}(\mathbf{w})$, $\underline{\mathbf{H}}$ is the $N \times N$ time-domain channel matrix, $\mathbf{F}$ is the $N \times N$ unitary DFT matrix, $\underline{\mathbf{a}}$ is the $N \times 1$ OFDM block that contains the frequency-domain data, $\underline{\mathbf{n}}_{\mathrm{W}} = \mathbf{F}\boldsymbol{\Delta}_{\mathrm{W}}\underline{\mathbf{v}}$ is the noise vector in the frequency domain, with $\underline{\mathbf{v}}$ the receiver AWGN vector in the time domain before windowing, with covariance $\sigma_v^2 \mathbf{I}_N$. In (1), we also define $\underline{\boldsymbol{\Lambda}} = \mathbf{F}\underline{\mathbf{H}}\mathbf{F}^H$, which is the frequency-domain channel matrix in the absence of windowing, and $\underline{\mathbf{C}}_{\mathrm{W}} = \mathbf{F}\boldsymbol{\Delta}_{\mathrm{W}}\mathbf{F}^H$, which is the circulant matrix representing the windowing operation in the frequency domain. These definitions represent a clear link with conventional OFDM, where windowing is absent. In this case, $\boldsymbol{\Delta}_{\mathrm{W}} = \mathbf{I}_N$ and hence $\underline{\mathbf{C}}_{\mathrm{W}} = \mathbf{I}_N$.

By assuming that N_{A} out of N subcarriers are active, the $N \times 1$ transmitted vector $\underline{\mathbf{a}}$ can be rewritten as

$$\underline{\mathbf{a}} = \mathbf{T}_{\mathrm{GB}}\mathbf{a} = [\mathbf{0}^T_{N_{\mathrm{V}}/2\times 1} \ \mathbf{a}^T \ \mathbf{0}^T_{N_{\mathrm{V}}/2\times 1}]^T , \tag{2}$$

where $\mathbf{T}_{\mathrm{GB}} = [\mathbf{0}^T_{N_{\mathrm{V}}/2\times N_{\mathrm{A}}} \ \mathbf{I}_{N_{\mathrm{A}}} \ \mathbf{0}^T_{N_{\mathrm{V}}/2\times N_{\mathrm{A}}}]^T$ is the $N \times N_{\mathrm{A}}$ matrix that inserts the N_{V} frequency guard bands, with $N_{\mathrm{V}} = N - N_{\mathrm{A}}$, and $\mathbf{a}$ is the $N_{\mathrm{A}} \times 1$ QPSK data vector with covariance $\sigma_a^2 \mathbf{I}_{N_{\mathrm{A}}}$. At the receiver side, we assume that the data received on the guard bands are discarded, as expressed by

$$\mathbf{z}_{\mathrm{W}} = \mathbf{R}_{\mathrm{GB}}\underline{\mathbf{z}}_{\mathrm{W}} = \boldsymbol{\Lambda}_{\mathrm{W}}\mathbf{a} + \mathbf{n}_{\mathrm{W}} , \tag{3}$$

where $\mathbf{R}_{\mathrm{GB}} = \mathbf{T}^T_{\mathrm{GB}}$ selects only the N_{A} middle subcarriers, $\boldsymbol{\Lambda}_{\mathrm{W}} = \mathbf{R}_{\mathrm{GB}}\underline{\boldsymbol{\Lambda}}_{\mathrm{W}}\mathbf{T}_{\mathrm{GB}}$ is $N_{\mathrm{A}} \times N_{\mathrm{A}}$, and $\mathbf{n}_{\mathrm{W}} = \mathbf{R}_{\mathrm{GB}}\underline{\mathbf{n}}_{\mathrm{W}}$ is $N_{\mathrm{A}} \times 1$.

Due to the time variation of the channel, the frequency-domain channel matrix $\underline{\boldsymbol{\Lambda}}_{\mathrm{W}}$ in (1) (or equivalently its $N_{\mathrm{A}} \times N_{\mathrm{A}}$ middle block $\boldsymbol{\Lambda}_{\mathrm{W}}$ in (3)) is not diagonal, but nearly banded [7], and each diagonal is associated with a discrete Doppler frequency that introduces ICI. Banded equalizers exploit this Doppler structure by approximating the channel matrix $\underline{\boldsymbol{\Lambda}}_{\mathrm{W}}$ with its banded version $\underline{\mathbf{B}}_{\mathrm{W}}$ (or equivalently $\boldsymbol{\Lambda}_{\mathrm{W}}$ with $\mathbf{B}_{\mathrm{W}}$), neglecting the ICI that comes from faraway subcarriers. We denote with Q the number of the relevant subcarriers from each side. The total bandwidth of $\underline{\mathbf{B}}_{\mathrm{W}}$ is hence $2Q+1$. If windowing is adopted, the receiver window can be designed to strengthen the banded assumption. This leads to the minimum band approximation error (MBAE) design of [8], where the window design minimizes the quantity $E\{\| \underline{\boldsymbol{\Lambda}}_{\mathrm{W}} - \underline{\mathbf{B}}_{\mathrm{W}} \|^2\}$ by exploiting the knowledge of the maximum Doppler frequency f_{D} and of the Doppler spectrum.

In the next section, we illustrate how to evaluate the BER performance of banded linear equalizers designed under the MMSE criterion. For an LE, the soft version of the data vector can be recovered by a matrix multiplication, expressed by

$$\tilde{\mathbf{a}}_{\mathrm{LE}} = \mathbf{G}\mathbf{z}_{\mathrm{W}} . \tag{4}$$

In case of windowing, the banded MMSE LE is expressed by [8]

$$\mathbf{G}_{\text{W-LE}} = \mathbf{B}^H_{\mathrm{W}}(\mathbf{B}_{\mathrm{W}}\mathbf{B}^H_{\mathrm{W}} + \gamma^{-1}\mathbf{R}_{\mathrm{GB}}\underline{\mathbf{C}}_{\mathrm{W}}\underline{\mathbf{C}}^H_{\mathrm{W}}\mathbf{R}^T_{\mathrm{GB}})^{-1} , \tag{5}$$

where $\gamma = \sigma_a^2 / \sigma_v^2$ represents the signal-to-noise (SNR) ratio. When windowing is absent, equation (5) reduces to

$$\mathbf{G}_{\mathrm{LE}} = \mathbf{B}^H(\mathbf{B}\mathbf{B}^H + \gamma^{-1}\mathbf{I}_{N_{\mathrm{A}}})^{-1} , \tag{6}$$

where $\mathbf{B}$ denotes the banded version of the $N_{\mathrm{A}} \times N_{\mathrm{A}}$ unwindowed channel matrix $\boldsymbol{\Lambda} = \mathbf{R}_{\mathrm{GB}}\underline{\boldsymbol{\Lambda}}\mathbf{T}_{\mathrm{GB}}$.

3. BER OF BANDED LINEAR EQUALIZERS

To derive the BER performance, we follow a semi-analytical approach that consists in two steps. First, assuming a given channel realization, we analytically derive the BER conditioned on that channel realization. Second, we average the conditional BER over the channel statistics. In both steps we will adopt some convenient approximations that permit to simplify the BER analysis. As a consequence of these approximations, the obtained BER is not truly exact. However, we will use computer simulations to establish the accuracy of the proposed approach. Although we focus on the performance of QPSK, the proposed semi-analytical method can be employed also for other symbol constellations.

For simplicity, let us consider the case when receiver windowing is not used. (Similar considerations holds for the windowing case.) By combining (4), (6), and (3), we obtain

$$\tilde{\mathbf{a}}_{\mathrm{LE}} = \mathbf{B}^H(\mathbf{B}\mathbf{B}^H + \gamma^{-1}\mathbf{I}_{N_{\mathrm{A}}})^{-1}(\boldsymbol{\Lambda}\mathbf{a} + \mathbf{n}) , \tag{7}$$

where $\boldsymbol{\Lambda} = \mathbf{R}_{\mathrm{GB}}\mathbf{F}\underline{\mathbf{H}}\mathbf{F}^H\mathbf{T}_{\mathrm{GB}}$ and $\mathbf{n} = \mathbf{R}_{\mathrm{GB}}\mathbf{F}\underline{\mathbf{v}}$ are the unwindowed versions of $\boldsymbol{\Lambda}_{\mathrm{W}}$ and $\mathbf{n}_{\mathrm{W}}$, respectively. Since $\mathbf{B}$ is the banded version of $\boldsymbol{\Lambda}$, we rewrite $\boldsymbol{\Lambda} = \mathbf{B} + \mathbf{E}$. It is clear that the matrix $\mathbf{E}$, which represents the ICI coming from the faraway subcarriers, has nonzero elements only outside its main band that contains $2Q+1$ diagonals. Since $\mathbf{E}$ generates an ICI that is left uncompensated by the equalizer, its effect should be considered as an additional source of error, as it happens for the noise. By inserting $\boldsymbol{\Lambda} = \mathbf{B} + \mathbf{E}$ in (7), we obtain

$$\tilde{\mathbf{a}}_{\mathrm{LE}} = \mathbf{B}^H(\mathbf{B}\mathbf{B}^H + \gamma^{-1}\mathbf{I}_{N_{\mathrm{A}}})^{-1}\mathbf{B}\mathbf{a} + \mathbf{G}_{\mathrm{LE}}\mathbf{u} , \tag{8}$$

where $\mathbf{u} = \mathbf{E}\mathbf{a} + \mathbf{n}$ represents the *aggregate noise* due to both the neglected ICI and the thermal AWGN. In (8), we observe that the matrix $\mathbf{B}^H(\mathbf{B}\mathbf{B}^H + \gamma^{-1}\mathbf{I}_{N_{\mathrm{A}}})^{-1}\mathbf{B} = (\mathbf{B}^H\mathbf{B} + \gamma^{-1}\mathbf{I}_{N_{\mathrm{A}}})^{-1}\mathbf{B}^H\mathbf{B}$ is not diagonal. This is a direct consequence of the MMSE criterion, which does not suppress the ICI completely, but allows some residual ICI in order to avoid excessive noise enhancement. Therefore, we define the matrix $\mathbf{D}$ as the diagonal matrix obtained by selecting the main diagonal of $\mathbf{B}^H(\mathbf{B}\mathbf{B}^H + \gamma^{-1}\mathbf{I}_{N_{\mathrm{A}}})^{-1}\mathbf{B}$, and the matrix

$$\boldsymbol{\Omega} = \mathbf{B}^H(\mathbf{B}\mathbf{B}^H + \gamma^{-1}\mathbf{I}_{N_{\mathrm{A}}})^{-1}\mathbf{B} - \mathbf{D} . \tag{9}$$

By (9), Equation (8) becomes

$$\tilde{\mathbf{a}}_{\mathrm{LE}} = \mathbf{D}\mathbf{a} + \boldsymbol{\Omega}\mathbf{a} + \mathbf{G}_{\mathrm{LE}}\mathbf{u} , \tag{10}$$

where $\mathbf{D}\mathbf{a}$ is the useful term, $\boldsymbol{\Omega}\mathbf{a}$ represents the residual ICI, and $\mathbf{G}_{\mathrm{BLE}}\mathbf{u}$ stands for the aggregate noise. Focusing on the *k*th subcarrier, the decision variable can be expressed by

$$\tilde{a}_k = [\tilde{\mathbf{a}}_{\mathrm{LE}}]_k = d_{k,k}a_k + \sum_{l\neq k,\, l=1}^{N_{\mathrm{A}}} \omega_{k,l}a_l + \sum_{l=1}^{N_{\mathrm{A}}} g_{k,l}u_l , \tag{11}$$

where a_l is the *l*th element of the vector $\mathbf{a}$, $\omega_{k,l}$ is the (*k*,*l*)-th element of $\boldsymbol{\Omega}$, and the other quantities are defined accordingly.

To obtain the conditional BER, the key idea is to approximate as Gaussian the undesired terms in (11). Indeed, the first summation in (11), expressed by

$$\beta_k = \sum_{l\neq k,\, l=1}^{N_{\mathrm{A}}} \omega_{k,l}a_l , \tag{12}$$

is the weighted sum of $N_{\mathrm{A}} - 1$ independent and equiprobable random variables $\{a_l\}$. Since in OFDM systems the number of active subcarriers is quite large (usually $N_{\mathrm{A}} \gg 10$), the hypotheses of the central limit theorem (CLT) [11] are satisfied, and therefore β_k in (12) can be safely considered as a zero-mean complex circular Gaussian random variable, with variance equal to

$$\sigma^2_{\beta_k} = \sigma_a^2 \sum_{l\neq k,\, l=1}^{N_{\mathrm{A}}} |\, \omega_{k,l} \,|^2 . \tag{13}$$

A more complete theoretical justification of this Gaussian approximation can be found, in the context of MMSE multiuser detection, in [12].

To statistically characterize the second summation in (11), we observe that u_l in (11), expressed by

$$u_l = n_l + \sum_{|k-l|>Q,\ l=1}^{N_A} e_{k,l} a_l \ , \qquad (14)$$

is obtained by adding to the Gaussian random variable n_l the weighted sum of $N_A - 2Q - 1$ independent and equiprobable random variables $\{a_l\}$. Practical values of Q are very small [8], and therefore, similarly to the previous case, the number of summed elements is quite large, and we can apply the CLT to approximate the summation in (14) as Gaussian. Consequently, u_l can be approximated as Gaussian as well, being the sum of two Gaussian random variables. More precisely, we approximate the whole vector $\mathbf{u} = \mathbf{Ea} + \mathbf{n}$ as a complex jointly Gaussian random vector with zero mean and covariance matrix $\mathbf{C}_{\mathbf{uu}}$ expressed by

$$\mathbf{C}_{\mathbf{uu}} = \sigma_a^2 \mathbf{E}\mathbf{E}^H + \sigma_n^2 \mathbf{I}_{N_A} \ . \qquad (15)$$

Since the aggregate noise in (11), expressed by

$$\theta_k = \sum_{l=1}^{N_A} g_{k,l} u_l \ , \qquad (16)$$

is a linear combination of random variables that are approximately jointly Gaussian, θ_k can be even better approximated as complex Gaussian. Its variance $\sigma_{\theta_k}^2$ is calculated as the (*k*,*k*)-th element of the matrix $\mathbf{G}_{\mathrm{LE}} \mathbf{C}_{\mathbf{uu}} \mathbf{G}_{\mathrm{LE}}^H$, which by (15) can be expressed as

$$\sigma_{\theta_k}^2 = \sigma_a^2 [\mathbf{G}_{\mathrm{LE}} \mathbf{E}\mathbf{E}^H \mathbf{G}_{\mathrm{LE}}^H]_{k,k} + \sigma_n^2 [\mathbf{G}_{\mathrm{LE}} \mathbf{G}_{\mathrm{LE}}^H]_{k,k} \ . \qquad (17)$$

Taking into account that β_k and θ_k are approximately Gaussian, and assuming QPSK with Gray coding, the BER on the *k*th subcarrier conditioned on a given channel realization $\mathbf{\Lambda}$ can be obtained as [13]

$$BER_{\mathrm{LE}}(k, \mathbf{\Lambda}) = Q\left(\sqrt{\frac{|d_{k,k}|^2 \sigma_a^2}{\sigma_{\beta_k}^2 + \sigma_{\theta_k}^2}} \right) , \qquad (18)$$

where $\sigma_{\beta_k}^2$ and $\sigma_{\theta_k}^2$ are expressed by (13) and (17), respectively. The final BER is then obtained as the average of (18) over the channel statistics and over the subcarrier index, as expressed by

$$BER_{\mathrm{LE}} = \frac{1}{N_A} \sum_{k=1}^{N_A} \int BER_{\mathrm{LE}}(k, \mathbf{\Lambda}) f_{\mathbf{\Lambda}}(\mathbf{\Lambda}) d\mathbf{\Lambda} \ , \qquad (19)$$

where $f_{\mathbf{\Lambda}}(\mathbf{\Lambda})$ is the probability density function of the channel realization $\mathbf{\Lambda}$. The analytical solution of the integral in (19) seems quite difficult, even for specific channel statistics (such as Rayleigh fading with Jakes' Doppler spectrum). Therefore, we approximate the integral in (19) by averaging (18) over a finite number H of computer-generated channel realizations $\{\mathbf{\Lambda}^{(j)}\}$, as

$$BER_{\mathrm{LE}} = \frac{1}{N_A H} \sum_{k=1}^{N_A} \sum_{j=1}^{H} BER_{\mathrm{LE}}(k, \mathbf{\Lambda}^{(j)}) \ . \qquad (20)$$

Although this semi-analytical approach does not give a closed-form solution, it can be adopted with a large variety of channel statistics. In addition, the simulation results presented in Sect. 5 show that a relatively-small number of channel realizations, e.g. $H = 1000$, produces a very good approximation.

We point out that the proposed method can be used also to evaluate the BER of other linear equalizers. For example, the linear MMSE equalizer presented in [6], which considers all the ICI, can be obtained by selecting $Q = N_A$. In this case, since $\mathbf{E} = \mathbf{0}$, the term θ_k in (16) vanishes. As a second example, a very-low-complexity MMSE LE can be obtained by selecting $Q = 0$, which renders the equalizer $\mathbf{G}$ in (6) diagonal. For QPSK and other constant-modulus constellations, this equalizer is equivalent to the conventional one-tap equalizer (up to a real subcarrier-dependent scaling factor that does not affect BER performance).

4. BER BOUNDS FOR BANDED DFE

In this section, we investigate on the BER of banded DFE designed following the MMSE criterion. The DFE use two filters: a feedforward filter $\mathbf{F}_{\mathrm{F}}$, and a feedback filter $\mathbf{F}_{\mathrm{B}}$. The soft-detected data is expressed by

$$\tilde{\mathbf{a}}_{\mathrm{DFE}} = \mathbf{F}_{\mathrm{F}} \mathbf{z}_{\mathrm{W}} - \mathbf{F}_{\mathrm{B}} \hat{\mathbf{a}}_{\mathrm{DFE}} \ , \qquad (21)$$

where $\hat{\mathbf{a}}_{\mathrm{DFE}}$ is the hard-detected data vector. Hence, the feedback filter $\mathbf{F}_{\mathrm{B}}$ cancels out the ICI generated from the already-detected data, whereas the feedforward filter $\mathbf{F}_{\mathrm{F}}$ suppresses the ICI coming from the not-yet-detected data. For banded MMSE DFE [8],

$$\mathbf{F}_{\mathrm{B}} = \mathbf{L}^H - \mathbf{I}_{N_A} \ , \qquad \mathbf{F}_{\mathrm{F}} = \mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \ , \qquad (22)$$

where $\mathbf{L}$ is a band lower triangular matrix obtained from band LDL decomposition of $\mathbf{M} = \gamma^{-1} \mathbf{I}_{N_A} + \mathbf{B}^H \mathbf{B}$, as expressed by $\mathbf{M} = \mathbf{LDL}^H$. (In case of windowing, $\mathbf{G}_{\text{W-LE}}$ of (5) and the LDL of $\mathbf{M}_{\mathrm{W}} = \gamma^{-1} \mathbf{I}_{N_A} + \mathbf{T}_{\mathrm{GB}}^T \mathbf{B}^H \mathbf{B} \mathbf{T}_{\mathrm{GB}}$ are used.)

Since DFE make use of past decisions cancellation, they suffer from error propagation. Therefore, an exact BER analysis is rather cumbersome, since should take into account all the possible error events. For banded DFE with QPSK, since $\mathbf{L}$ is banded with upper band equal to $2Q$, the number of possible error events is 9^{2Q} [9], which is tractable only for $Q = 1$. Although we are currently looking for convenient approximations of the exact BER, for simplicity, we present herein some simple bounds that could be useful to predict the BER behavior. We describe these bounds for the unwindowed case (the windowed case is analogous).

4.1. Lower Bound

A simple BER lower bound can be obtained by assuming $\hat{\mathbf{a}}_{\mathrm{DFE}} = \mathbf{a}$ in (21), i.e., we neglect the error propagation. In this case, from (3), (21)-(22), we obtain

$$\tilde{\mathbf{a}}_{\mathrm{DFE}} = (\mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \mathbf{\Lambda} - \mathbf{L}^H + \mathbf{I}_{N_A}) \mathbf{a} + \mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \mathbf{n} \ , \qquad (23)$$

which, by splitting $\mathbf{\Lambda} = \mathbf{B} + \mathbf{E}$, and using $\mathbf{u} = \mathbf{Ea} + \mathbf{n}$, becomes

$$\tilde{\mathbf{a}}_{\mathrm{DFE}} = \mathbf{D}_{\mathrm{DFE}} \mathbf{a} + \mathbf{\Omega}_{\mathrm{DFE}} \mathbf{a} + \mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \mathbf{u} \ , \qquad (24)$$

where $\mathbf{D}_{\mathrm{DFE}}$ is the diagonal matrix obtained by selecting the main diagonal of the matrix $\mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \mathbf{B} - \mathbf{L}^H + \mathbf{I}_{N_A}$, and $\mathbf{\Omega}_{\mathrm{DFE}} = \mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \mathbf{B} - \mathbf{L}^H + \mathbf{I}_{N_A} - \mathbf{D}_{\mathrm{DFE}}$. It is clear that (24) is the DFE counterpart of (10). Similarly to the BER of the LE, we can apply the Gaussian approximation to the interference terms $\mathbf{\Omega}_{\mathrm{DFE}} \mathbf{a}$ and $\mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \mathbf{u}$. By omitting a detailed derivation, which is similar to that in Sect. 3, we obtain the lower bound on the BER for subcarrier *k* as

$$BER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda}) = Q\left(\sqrt{\frac{|d_{\mathrm{DFE},k,k}|^2 \sigma_a^2}{\sigma_{\mathrm{DFE},\beta_k}^2 + \sigma_{\mathrm{DFE},\theta_k}^2}} \right) , \qquad (25)$$

where $d_{\mathrm{DFE},k,k}$ is the (*k*,*k*)-th element of the matrix $\mathbf{D}_{\mathrm{DFE}}$, and

$$\sigma_{\mathrm{DFE},\beta_k}^2 = \sigma_a^2 \sum_{l \neq k,\ l=1}^{N_A} |\omega_{\mathrm{DFE},k,l}|^2 \ , \qquad (26)$$

$$\sigma_{\mathrm{DFE},\theta_k}^2 = \sigma_a^2 [\mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \mathbf{E}\mathbf{E}^H \mathbf{G}_{\mathrm{LE}}^H \mathbf{L}]_{k,k} + \sigma_n^2 [\mathbf{L}^H \mathbf{G}_{\mathrm{LE}} \mathbf{G}_{\mathrm{LE}}^H \mathbf{L}]_{k,k} \ . \qquad (27)$$

The final lower bound can then be obtained as in (20) by

$$BER_{\mathrm{DFE}}^{(\mathrm{LB})} = \frac{1}{N_A H} \sum_{k=1}^{N_A} \sum_{j=1}^{H} BER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda}^{(j)}) \ . \qquad (28)$$

4.2. Approximate Upper Bound

To derive the BER upper bound, we consider the error propagation following the approach of [10]. For each subcarrier k, since the data are detected sequentially from $k = N_\mathrm{A}$ to $k = 1$, only $M_k = N_\mathrm{A} - k$ previously-detected symbols can propagate one or more possible errors. Hence, for each subcarrier k, we introduce $M_k + 1$ states, identified by ϕ_m, with $0 \le m \le M_k$. We say that the state of subcarrier k is ϕ_m when the closest error has happened on subcarrier $k+m+1$. In other words, the state is ϕ_m when all the previous symbols from $k+1$ to $k+m$ are detected correctly, while the symbol $k+m+1$ is incorrect. Specifically, the state ϕ_0 identifies a symbol error on the previously-detected subcarrier (with index $k+1$), while the state ϕ_{M_k} represents the absence of errors on all the previously-detected symbols. We also define the quantity $p_{k,m}$ as the probability that the state of subcarrier k is ϕ_m, and the conditional probability of correct decision as

$$\alpha_{k,m} = \Pr\{\hat{a}_k = a_k \mid \phi_m\} . \tag{29}$$

These quantities allow a complete description of the state-transition diagram. Specifically, for $0 \le m < M_k$, the system can evolve from the state ϕ_m only towards two states: either ϕ_{m+1} (with probability $\alpha_{k,m}$) or ϕ_0 (with probability $1-\alpha_{k,m}$). The state-transition equations can be expressed by

$$p_{k-1,0} = \sum_{m=0}^{M_k} (1-\alpha_{k,m}) p_{k,m} , \tag{30}$$

$$p_{k-1,m} = \alpha_{k,m-1} p_{k,m-1} , \qquad 1 \le m \le M_k - 1 , \tag{31}$$

$$p_{k-1,M_k} = \alpha_{k,M_k-1} p_{k,M_k-1} + \alpha_{k,M_k} p_{k,M_k} . \tag{32}$$

Due to the channel correlation in the frequency domain, we expect that the conditional symbol error rate (SER) on two adjacent subcarriers is roughly equal. Therefore, by recalling that ϕ_0 identifies a symbol error on the previously-detected subcarrier, we can write

$$SER_{\mathrm{DFE}}(k, \mathbf{\Lambda}) = p_{k-1,0} \approx SER_{\mathrm{DFE}}(k+1, \mathbf{\Lambda}) = p_{k,0} . \tag{33}$$

By the same reason, we impose the stationary approximation $p_{k-1,m} \approx p_{k,m}$ also for $m \ne 0$. It is obvious that this assumption can not be strictly true, since the number of states depends on k. However, by this approximation, we are able to apply (30)-(32) recursively, in order to explicit the conditional SER as [10]

$$SER_{\mathrm{DFE}}(k, \mathbf{\Lambda}) = \left(1 + \sum_{i=0}^{M_k-2} \prod_{m=0}^{i} \alpha_{k,m} + \frac{1}{1-\alpha_{k,M_k}} \prod_{m=0}^{M_k-1} \alpha_{k,m} \right)^{-1} . \tag{34}$$

At this point, a SER upper bound can be simply obtained by fixing some specific values $\bar{\alpha}_{k,m}$, with $\bar{\alpha}_{k,m} \le \alpha_{k,m}$, thereby increasing the conditional SER. Since we are dealing with Gray coding, we can neglect the possibility that a symbol error will produce more than one bit error. This is equivalent of assuming that only three symbols out of four are possible. Hence, we can select

$$\bar{\alpha}_{k,m} = 1/3 , \qquad 0 \le m \le 2Q - 1 . \tag{35}$$

In addition, for the banded DFE, the error propagation is limited to $2Q$ symbols, and hence we can write

$$\bar{\alpha}_{k,m} = \alpha_{k,m} = 1 - SER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda}) , \qquad 2Q \le m \le M_k , \tag{36}$$

where $SER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda})$ is the SER in the absence of error propagation, which can be calculated from (25) as

$$SER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda}) = 2\, BER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda}) - \left(BER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda}) \right)^2 . \tag{37}$$

From (34), by using (35) and (36), we can obtain

$$SER_{\mathrm{DFE}}^{(\mathrm{UB})}(k, \mathbf{\Lambda}) = \frac{2 \cdot 3^{2Q} SER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda})}{3 \cdot (3^{2Q} - 1) SER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda}) + 2} \ge SER_{\mathrm{DFE}}(k, \mathbf{\Lambda}) . \tag{38}$$

This explains that, for low SER values, the SER is at maximum 3^{2Q} times higher than the SER in the absence of error propagation. Although we are interested in very small values of Q, such as $Q = 1$ and $Q = 2$, this upper bound seems too loose for our purposes. Indeed, in OFDM systems, the ICI coefficients are very small. This translates into a feedback filter with very small coefficients with respect to the dominant one, which represents the cancellation of the adjacent symbol. As a consequence, we re-derive the upper bound by imposing (36) also for $1 \le m \le 2Q - 1$, while maintaining $\bar{\alpha}_{k,0} = 1/3$. In this case, the approximate upper bound becomes

$$SER_{\mathrm{DFE}}^{(\mathrm{AUB})}(k, \mathbf{\Lambda}) = \frac{3\, SER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda})}{3\, SER_{\mathrm{DFE}}^{(\mathrm{LB})}(k, \mathbf{\Lambda}) + 1} . \tag{39}$$

The final upper bound can then be obtained as in (20) and (28) by

$$BER_{\mathrm{DFE}}^{(\mathrm{AUB})} = \frac{1}{2 N_\mathrm{A} H} \sum_{k=1}^{N_\mathrm{A}} \sum_{j=1}^{H} SER_{\mathrm{DFE}}^{(\mathrm{AUB})}(k, \mathbf{\Lambda}^{(j)}) , \tag{40}$$

where the factor 2 takes into account that, for QPSK with Gray coding, a symbol error generally translates into a single bit error.

5. SIMULATION RESULTS

We consider an OFDM system with $N = 128$, $N_\mathrm{A} = 96$, CP length $L = 8$, and QPSK with Gray coding. The channel model is characterized by Rayleigh fading, an exponential power-delay profile with root-mean-square delay spread $\sigma = 3$ (normalized to the sampling period), truncated up to the CP length, and a Jakes' Doppler spectrum. In case of windowing, we adopted the MBAE-SOE window design of [8]. All the theoretical results have been obtained by averaging over $H = 1000$ channel realizations.

We firstly compare the analytical and the simulated BER of the linear equalizers. Fig. 1 illustrates the BER of the MMSE LE as a function of the SNR per bit $\gamma/2$, when the normalized Doppler spread is $f_\mathrm{D}/\Delta_f = 0.15$, i.e., 15% of the subcarrier separation. It is evident that the theoretical BER coincides with the simulated one both in the presence or absence of windowing. Our semi-analytical approach is able to correctly predict also the BER of one-tap equalization, which is the conventional equalization scheme tailored to time-invariant channels. This agrees with the result in [4]. The proposed approach is useful also to characterize the BER performance of the full (i.e., non-banded) MMSE LE of [6]. Fig. 2 investigates the BER floors of linear equalizers, as a function of the normalized Doppler spread. The SNR per bit is fixed and equal to 40 dB. Also in this case the theoretical BER closely fits with the simulated one.

We now compare the simulated BER of the DFE with the theoretical BER bounds, i.e., the lower bound (LB) of (28) and the approximate upper bound (AUB) of (40). Figs. 3-4 show that the two bounds correctly follow the BER shape. Unfortunately, they are not very tight. This means that other refined approximations can potentially tighten the two bounds. A second possibility is to develop a BER analysis using the approach of [9]. We are currently working on both the alternatives.

6. CONCLUSIONS

We have described a semi-analytical method that predicts the BER of a banded MMSE LE in OFDM systems with significant Doppler spread. The comparison with simulation results has shown that the proposed approach is quite accurate and can be adopted also for other (non-banded) detectors. In addition, we have presented a lower bound and an approximate upper bound on the performance of banded MMSE DFE. Future work could improve the DFE error propagation model, and incorporate the effect of channel coding on the BER performance.

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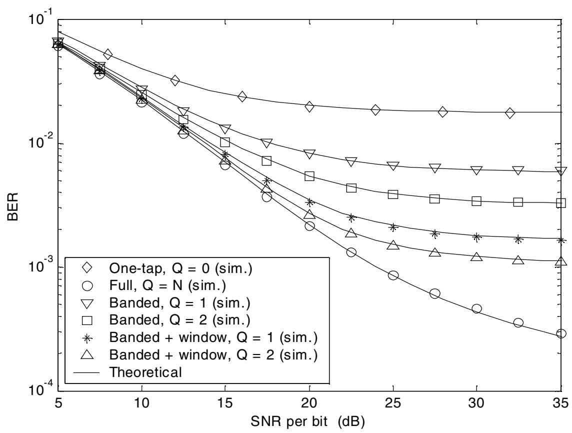

Fig. 1. BER of linear equalizers ($f_D/\Delta_f = 0.15$).

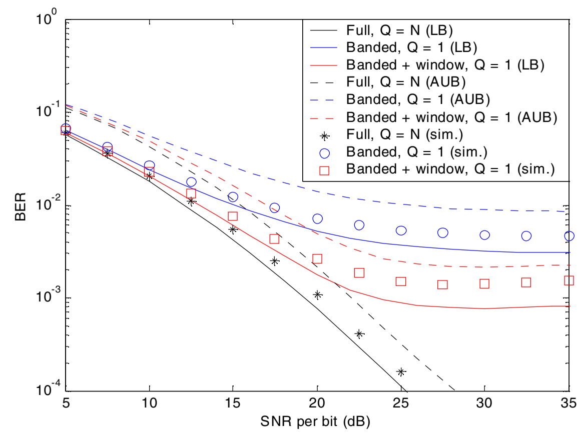

Fig. 3. BER of decision-feedback equalizers.

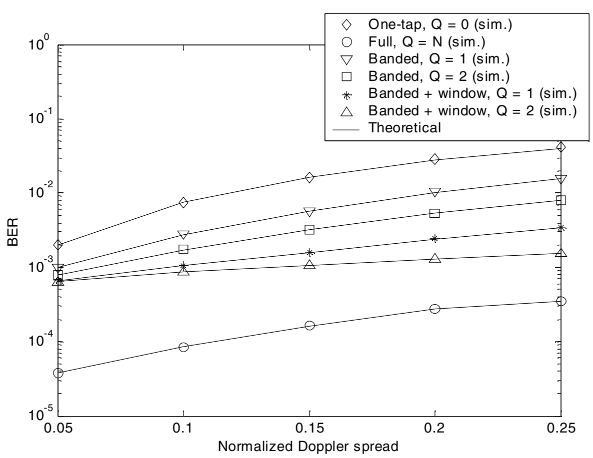

Fig. 2. BER of linear equalizers ($\gamma/2 = 40$ dB).

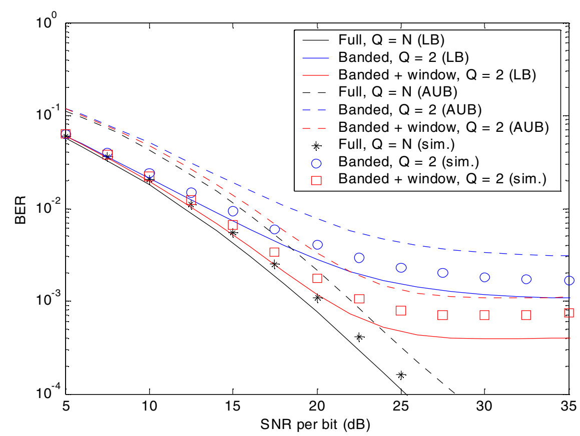

Fig. 4. BER of decision-feedback equalizers.