A Full-Rank Regularization Technique for MMSE Detection in Multiuser CDMA Systems

Luca Rugini, *Member, IEEE,* Paolo Banelli, *Member, IEEE,* and Saverio Cacopardi, *Member, IEEE*

***Abstract*— In multiuser code-division multiple-access (CDMA) environments, the ill-conditioning of the covariance matrix of the received signal may degrade the performance of minimum mean-squared error (MMSE) detectors, especially when few samples are available for the covariance matrix estimation. In order to mitigate this performance degradation, we propose a full-rank regularized MMSE detector based on the covariance matrix tapering (CMT) technique. Simulation results show the effectiveness of the proposed technique at high signal-to-noise ratio (SNR).**

***Index Terms*— CDMA, covariance matrix tapering, MMSE, multiuser detection, regularization.**

I. Introduction

It is well known that the MMSE detector for CDMA systems can be implemented as a Wiener filter at the chip level, by multiplying the inverse of the covariance matrix of the received signal with the signature waveform of the user of interest [1]. In realistic scenarios, the covariance matrix may be affected by significant estimation errors. This happens when only a small sample set is available for the estimation, e.g., if the transmitted blocks are very short or if the channel coherence time is not so high. In addition, the covariance matrix can be ill-conditioned, with a high eigenvalue spread for high values of the SNR. As a consequence of the matrix inverse contained in the MMSE detector, the high eigenvalue spread greatly enhances the covariance matrix estimation errors.

In order to counteract such errors, a possible approach relies on regularization techniques [2], which improve the numerical conditioning by convenient modifications of the estimated covariance matrix. Regularization methods also include some reduced-rank techniques [1], which project the received signal onto a lower-dimensional subspace, thereby resulting in a covariance matrix with smaller eigenvalue spread.

We show herein that the covariance matrix tapering (CMT), a full-rank regularization technique proposed in [3][4] for beamforming applications, is effective also in multiuser CDMA scenarios. In this context, we introduce two new tapering matrix families. Simulation results show the performance gain obtained at high SNR by the proposed CMT approach.

Manuscript received April 1, 2004. The associate editor coordinating the review of this letter and approving it for publication was Prof. P. Cotae. This paper appeared in part at the Int. Conf. on Communications, Anchorage, AK, May 2003.

The authors are with D.I.E.I., University of Perugia, Italy (e-mail: rugini@diei.unipg.it)

Digital Object Identifier 10.1109/LCOMM.2005.01005.

II. MMSE Detection of CDMA Signals

A CDMA system with K active users and processing gain N is considered. After chip-rate sampling, the received vector relative to the lth transmitted symbol can be expressed as [5]

$$\mathbf{r}[l] = \mathbf{H}\mathbf{b}[l] + \mathbf{n}[l], \tag{1}$$

where $\mathbf{r}[l]$ is an MN-dimensional vector, M is the smoothing factor, $\mathbf{H}$ is the $MN \times K(L+M)$ block Toeplitz channel matrix containing the signature waveforms of all the users, $\mathbf{b}[l] = [b_1[l-L], ..., b_K[l-L], ..., b_1[l+M-1], ..., b_K[l+M-1]]^T$ is the $(KL+KM)$-dimensional vector that contains the data symbols of all the users, L is the length of the time-dispersive channel in symbol intervals, and $\mathbf{n}[l]$ represents the additive white Gaussian noise (AWGN) with covariance $\sigma^2\mathbf{I}_{MN}$. The multipath channel of each user is assumed time-invariant over the transmission of P consecutive symbols.

Without loss of generality, we consider binary phase-shift keying (BPSK) modulation. The decision rule of a generic linear receiver is expressed by

$$\hat{b}_k[l] = \mathrm{sign}(\mathrm{Re}(\mathbf{w}_k^H \mathbf{r}[l])), \tag{2}$$

where $\mathbf{w}_k$ is the MN-dimensional vector that represents the detector of the user k. The detector that minimizes the mean-squared error $E\{|b_k[l] - \mathbf{w}_k^H \mathbf{r}[l]|^2\}$ can be expressed by [1]

$$\mathbf{w}_{\mathrm{MMSE},k} = \mathbf{R}^{-1}\mathbf{h}_k, \tag{3}$$

where $\mathbf{R} = E\{\mathbf{r}[l]\mathbf{r}[l]^H\} = \mathbf{H}\mathbf{H}^H + \sigma^2\mathbf{I}_{MN}$ is the $MN \times MN$ covariance matrix of $\mathbf{r}[l]$, and $\mathbf{h}_k$, which is the $(KL+k)$th column of $\mathbf{H}$ [5], is the signature waveform of the user k. The estimated version of the MMSE receiver is expressed by

$$\hat{\mathbf{w}}_{\mathrm{SMI},k} = \hat{\mathbf{R}}^{-1}\hat{\mathbf{h}}_k, \tag{4}$$

where $\hat{\mathbf{R}}^{-1}$ is the sample matrix inverse (SMI). Thus

$$\hat{\mathbf{R}} = \frac{1}{P}\sum_{l=0}^{P-1} \mathbf{r}[l]\mathbf{r}[l]^H \tag{5}$$

is obtained without explicit knowledge of $\mathbf{H}$ and of σ^2, and the estimated signature waveform $\hat{\mathbf{h}}_k$ is obtained exploiting training sequences or blind channel estimation techniques [6].

In many practical situations, the value of P in (5) has to be kept small. As an example, when $\rho = P/(MN)$ is smaller than 1, $\hat{\mathbf{R}}$ is not full rank and therefore not invertible. As a rule of thumb, we can assume that the covariance estimation errors are small when $\rho > 6$ [7]. However, when $K < N$, $\mathbf{H}$ can be tall, with $D = \mathrm{rank}(\mathbf{H})$ strictly lower than MN. In such a situation, $\mathbf{R}$ is often ill-conditioned, especially when the SNR is high, and hence the matrix inversion in (4) causes significant performance degradation.

It should be pointed out that the problems induced by a small P are exacerbated when multiple antennas are used at the receiver side, because the dimension of $\mathbf{R}$ increases by a factor equal to the number of the receiving antennas.

III. Regularized MMSE Detectors

Regularization techniques [2] deal with ill-conditioned problems by substituting the matrix $\mathbf{R}$ with a matrix characterized by a smaller eigenvalue spread. Consequently, the variance of the estimation errors decreases, at the cost of introducing some bias in the detector estimate. The goal is to find a good trade-off between bias and variance.

A. Regularization by Covariance Matrix Loading

Although not widely recognized, some regularization techniques have already been used in multiuser detection. Indeed, the constrained minimum output energy (CMOE) receiver of [6] exploits a particular form of Tikhonov regularization [2] by replacing the matrix $\mathbf{R}$ with $\mathbf{R} + \nu\mathbf{I}_{MN}$, as expressed by

$$\hat{\mathbf{w}}_{\mathrm{CMOE},k} = (\hat{\mathbf{R}} + \nu\mathbf{I}_{MN})^{-1}\hat{\mathbf{h}}_k, \tag{6}$$

where $\nu = \alpha\mathrm{tr}(\hat{\mathbf{R}})$ is a positive parameter. By using the eigenvalue decomposition (EVD) $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$, it is easy to verify the eigenvalue spread reduction. In the array processing literature, this detector is known as diagonal loaded SMI [7].

B. Regularization by Eigendecomposition Truncation

A different regularized detector can be obtained by applying the EVD to $\mathbf{R}$ and neglecting the eigenvectors associated with the $MN - r$ smallest eigenvalues. Using this kind of regularization, usually referred as truncated singular value decomposition [2] or principal components (PC) analysis, the reduced-rank detector of the user k can be expressed as

$$\hat{\mathbf{w}}_{\mathrm{PC},k} = \hat{\mathbf{U}}_r\hat{\mathbf{\Lambda}}_r^{-1}\hat{\mathbf{U}}_r^H\hat{\mathbf{h}}_k, \tag{7}$$

where $\mathbf{U}_r$ contains only the r selected eigenvectors. If r is equal to $D = \mathrm{rank}(\mathbf{H})$, the detector is constrained to lie in the signal subspace [5]. However, a choice $r < D$ could give better performance because of the reduced eigenvalue spread [2].

C. Regularization by Krylov Subspace Constraint

A regularizing effect can also be obtained by constraining the detector to lie in the r-dimensional Krylov subspace $\mathrm{K}_r(\mathbf{R}, \mathbf{h}_k) = \mathrm{span}\{\mathbf{h}_k, \mathbf{R}\mathbf{h}_k, ..., \mathbf{R}^{r-1}\mathbf{h}_k\}$, because this solution can be considered as an approximation of the PC solution [2]. The MMSE detector within the Krylov subspace, known as multistage Wiener filter (MSWF) [8], can be expressed by

$$\hat{\mathbf{w}}_{\mathrm{MSWF},k} = \hat{\mathbf{V}}_{k,r}(\hat{\mathbf{V}}_{k,r}^H\hat{\mathbf{R}}\hat{\mathbf{V}}_{k,r})^{-1}\hat{\mathbf{V}}_{k,r}^H\hat{\mathbf{h}}_k, \tag{8}$$

where $\hat{\mathbf{V}}_{k,r}$ is an orthonormal basis of $\mathrm{K}_r(\hat{\mathbf{R}}, \hat{\mathbf{h}}_k)$.

D. Regularization by Covariance Matrix Tapering

We propose herein a new multiuser detector based on the CMT approach, which has been suggested for beamforming in order to widen the nulls of the antenna array pattern [3][4]. The idea of CMT is to multiply each element of $\mathbf{R}$ with a different weight, attenuating those elements far apart from the main diagonal. The CMT detector can be expressed as

$$\hat{\mathbf{w}}_{\mathrm{CMT},k} = (\hat{\mathbf{R}} \circ \mathbf{T})^{-1}\hat{\mathbf{h}}_k, \tag{9}$$

where the symbol $\circ$ represents the Hadamard (element-wise) product [9], and $\mathbf{T}$ is the tapering matrix. In the following, we show that the CMT approach gives rise to an eigenvalue spread $\chi(\mathbf{R} \circ \mathbf{T})$ smaller than $\chi(\mathbf{R})$.

Theorem 1: If $\mathbf{R}$ is an $MN \times MN$ positive definite matrix and $\mathbf{T}$ is an $MN \times MN$ correlation matrix [9], i.e., $\mathbf{T}$ is Hermitian positive semidefinite with $\mathbf{I}_{MN} \circ \mathbf{T} = \mathbf{I}_{MN}$, then

$$\chi(\mathbf{R} \circ \mathbf{T}) \leq \chi(\mathbf{R}). \tag{10}$$

Proof: See the Appendix. ■

Theorem 1 proves that the CMT certainly is a full-rank regularization technique, but it does not guide us in the design of a suitable tapering matrix. In [3], $\mathbf{T}$ is chosen as

$$[\mathbf{T}_{\mathrm{sinc},\alpha}]_{m,n} = \mathrm{sinc}(\alpha|m-n|) = \frac{\sin(\pi\alpha|m-n|)}{\pi\alpha|m-n|}, \tag{11}$$

where $\alpha > 0$ is the regularization parameter. Anyway, by Theorem 1, any correlation matrix can be selected. For instance, the CMOE receiver can be interpreted as a way to enhance the main diagonal of $\mathbf{R}$ with respect to all the other diagonals. However, since multipath channels introduce significant correlation between nearby chips, also the diagonals close to the main diagonal should be enhanced with respect to the faraway diagonals, thus motivating the CMT approach. Hence, a reasonable choice of $\mathbf{T}$ should produce an attenuation that increases when moving away from the main diagonal.

Moreover, it should be pointed out that the sinc-shaped profile in (11) seems to be inappropriate for both high and low values of α. Indeed, for high values of α, the sinc function has an oscillating behavior, and some weights are negative. On the other hand, for low α, the sinc function is concave. This implies that several diagonals close to the main diagonal are weighted with similar weights, while only few faraway diagonals are attenuated. This philosophy is just the opposite with respect to the one of the well known CMOE detector.

As a result of the previous remarks, we propose a matrix $\mathbf{T}$ whose tapering profile has a second derivative equal to zero (i.e., the diagonals are weighted linearly), as expressed by

$$[\mathbf{T}_{\mathrm{tri},\alpha}]_{m,n} = \mathrm{clip}(1 - \alpha|m-n|), \tag{12}$$

where the clipping function $\mathrm{clip}(x)$ forces the negative values of x to zero. Furthermore, we also propose a matrix $\mathbf{T}$ whose profile is a convex function, as expressed by

$$[\mathbf{T}_{\exp,\alpha}]_{m,n} = e^{-\alpha|m-n|}. \tag{13}$$

In all the cases, the parameter $\alpha > 0$ controls the amount of regularization. As it happens for the CMOE receiver, the CMT receiver is equivalent to the SMI receiver when the regularization parameter α is set to zero, while it tends to the RAKE receiver for high values of α. Indeed, for increasing

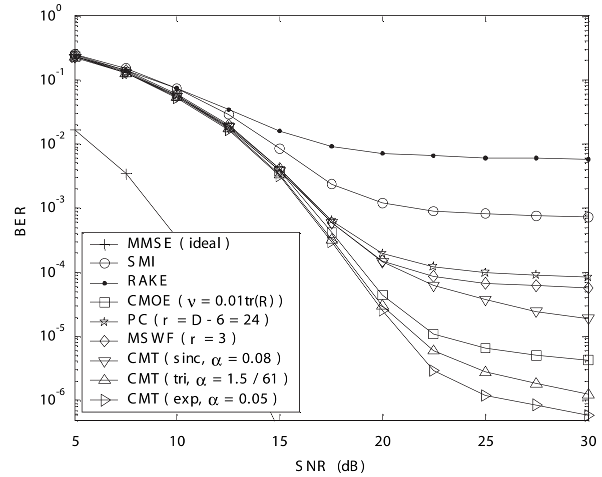

Fig. 1. BER comparison of various detectors as a function of the SNR.

α, the tapering matrices defined by (12) and (13) tend to $\mathbf{I}_{MN}$. In this case, only the elements of the main diagonal of $\hat{\mathbf{R}}$ are selected, and, since these elements are nearly equal, the CMT receiver in (9) is practically a scaled version of $\hat{\mathbf{w}}_{\mathrm{RAKE,k}} = \hat{\mathbf{h}}_k$. When such an amount of regularization is applied, $\chi(\mathbf{R} \circ \mathbf{T})$ is roughly equal to 1, but the receiver has lost the interference mitigation capability of the MMSE receiver.

Of course, neither $\alpha = 0$ nor $\alpha \to +\infty$ are optimal in the short data record case. The optimum value of α depends not only on the chosen tapering matrix, but also on the scenario. Obviously, when ρ increases, the optimum value of α should decrease. Different algorithms for the automatic choice of α can be derived by exploiting the methods employed for other regularization techniques [2]. These algorithms are still under investigation and are not considered in the present work.

IV. Simulation Results

In this section, we compare by simulations the performance of the regularized detectors introduced so far. We consider a downlink situation with a base station that transmits data with equal power to $K = 10$ active users. Gold sequences of length $N = 31$ have been chosen for the short spreading codes. The amplitudes of the 15 chip-spaced channel paths are modeled as independent zero-mean complex Gaussian random variables with equal variance. Since the channel memory in symbol intervals is $L = 1$, the size of the receiving window has been fixed to $M = 2$. The length of the data block has been fixed to $P = 372$, leading to $\rho = 6$. The estimated signature waveform $\hat{\mathbf{h}}_k$ is obtained by using the subspace approach of [5].

Fig. 1 compares the BER of the different detectors averaged over 300 channel realizations. In order to have a fair comparison, we have chosen the regularization parameter that gives the best performance for each detector. In particular, the best value of α for the CMT detector turns out to be almost independent of the SNR [10]. It is noteworthy that the two proposed CMT detectors outperform the one with the sinc profile (11). Moreover, when SNR $>$ 15 dB, the CMT with exponential profile (13) gives the best performance among all the estimated detectors. In this specific simulated scenario, the full-rank detectors CMT and CMOE present lower BER with respect to the reduced-rank ones. This is probably due to the higher number of degrees of freedom offered by the MN-dimensional subspace. At lower SNR, significant BER differences are not recorded.

V. Conclusion

In this contribution, we have proposed a full-rank regularized MMSE detector for CDMA systems with small data sets. Two new tapering matrix families have been introduced. We have shown that the proposed CMT detector is quite effective at high SNR. The study of algorithms for the choice of the regularization parameter could be the subject of future work.

Appendix
Proof of Theorem 1

Theorem 2: [9, p. 338] If $\mathbf{R}$ is an $MN \times MN$ Hermitian matrix and $\mathbf{T}$ is an $MN \times MN$ correlation matrix, then

$$\sum_{i=1}^{n} \lambda_i(\mathbf{R} \circ \mathbf{T}) \le \sum_{i=1}^{n} \lambda_i(\mathbf{R}), \quad \forall n = 1, ..., MN, \tag{14}$$

where the eigenvalues $\{\lambda_i\}$ are ordered in decreasing order.

By using Theorem 2 with $n = 1$, we have

$$\lambda_{\max}(\mathbf{R} \circ \mathbf{T}) \le \lambda_{\max}(\mathbf{R}). \tag{15}$$

Moreover, by setting $n = MN$ in (14), we obtain

$$\sum_{i=1}^{MN} \lambda_i(\mathbf{R} \circ \mathbf{T}) = \mathrm{tr}(\mathbf{R} \circ \mathbf{T}) = \mathrm{tr}(\mathbf{R}) = \sum_{i=1}^{MN} \lambda_i(\mathbf{R}), \tag{16}$$

which jointly to (14) with $n = MN - 1$ leads to

$$\lambda_{\min}(\mathbf{R} \circ \mathbf{T}) \ge \lambda_{\min}(\mathbf{R}). \tag{17}$$

Therefore, by (15) and (17), we have

$$\chi(\mathbf{R} \circ \mathbf{T}) = \frac{\lambda_{\max}(\mathbf{R} \circ \mathbf{T})}{\lambda_{\min}(\mathbf{R} \circ \mathbf{T})} \le \frac{\lambda_{\max}(\mathbf{R})}{\lambda_{\min}(\mathbf{R})} = \chi(\mathbf{R}). \tag{18}$$

References

[1] M. Honig and M. K. Tsatsanis, “Adaptive techniques for multiuser CDMA,” *IEEE Signal Processing Mag.*, pp. 49-61, May 2000.

[2] P. C. Hansen, *Rank-Deficient and Discrete Ill-Posed Problems*, SIAM, 1998.

[3] M. Zatman, “Production of adaptive array troughs by dispersion synthesis,” *IEE Electron. Lett.*, vol. 31, pp. 2141-2142, 7 Dec. 1995.

[4] J. R. Guerci, “Theory and application of covariance matrix tapers for robust adaptive beamforming,” *IEEE Trans. Signal Processing*, vol. 47, pp. 977-985, Apr. 1999.

[5] X. Wang and H. V. Poor, “Blind equalization and multiuser detection in dispersive CDMA channels,” *IEEE Trans. Commun.*, vol. 46, pp. 91-103, Jan. 1998.

[6] U. Madhow, “Blind adaptive interference suppression for direct-sequence CDMA,” *Proc. IEEE*, vol. 86, pp. 2049-2069, Oct. 1998.

[7] B. D. Carlson, “Covariance matrix estimation errors and diagonal loading in adaptive arrays,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. 24, pp. 397-401, July 1988.

[8] M. L. Honig and J. S. Goldstein, “Adaptive reduced-rank interference suppression based on the multistage Wiener filter,” *IEEE Trans. Commun.*, vol. 50, pp. 986-994, June 2002.

[9] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, 1994.

[10] L. Rugini, P. Banelli, and S. Cacopardi, “Regularized MMSE multiuser detection using covariance matrix tapering,” *IEEE Int. Conf. Communications*, Anchorage, AK, vol. 4, pp. 2460-2464, May 2003.