

Bayesian Estimation of a Gaussian Source in Middleton's Class-A Impulsive Noise

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Abstract—This letter derives the minimum mean square error (MMSE) Bayesian estimator for a Gaussian source impaired by additive Middleton's Class-A impulsive noise. Additionally, as low-complex alternatives, the letter considers two popular suboptimal estimators, such as the soft-limiter and the blanker. The optimum MMSE thresholds for these suboptimal estimators are obtained by iteratively solving fixed point equations. The theoretical findings are corroborated by simulation results, which highlight the MSE performance penalty of the suboptimal estimators may be negligible with respect to the optimal Bayesian estimator (OBE). Noteworthy, the proposed estimators can be extended to any noise, or observation error, that can be modeled as a Gaussian-mixture.

Index Terms—Blanker, Gaussian-mixtures, impulsive noise, interference, Middleton's Class-A noise, MMSE estimation, soft-limiter.

I. INTRODUCTION

INTERFERENCE and noise with impulsive non-Gaussian distributions may impair the performance of several systems, including communications, controls, sensors and so forth. Middleton proposed widely accepted canonical models for interference, which characterize “intelligent” (e.g., information bearing), as well as “non-intelligent” (e.g., natural or man-made) noises [1]. Middleton's noise models were widely investigated to identify the interference behavior [1], [2], to estimate their canonical parameters [3], and to detect finite alphabets in digital communications [4], [5]. However, to the best of the author knowledge, the MMSE optimum Bayesian estimator (OBE) for a Gaussian source in Class-A impulsive noise, derived in this letter, is still lacking in the literature.

The use of the OBE may be restricted in some applications due to complexity constraints, especially if impulsive noise protection is granted by analogic equipments. In this case, simpler suboptimal devices, such as the soft-limiter (SL) or the blanking-nonlinearity (BN), may be used to clip or null out, respectively, the received signal when its magnitude overpasses a given threshold. For the two suboptimal estimators, the computation of the optimum thresholds (in the MMSE sense) can be formulated as a fixed-point problem [6], which always admits a solution by fast iterative approaches.

Practically, such OBE, SL-estimator (SLE), and BN-estimator (BNE) are helpful when the quantity of interest is modeled, or approximated, by a Gaussian probability density

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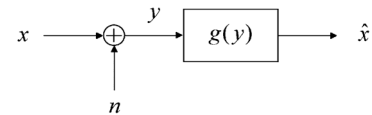


Fig. 1. System model.

function (*pdf*). This is the case for instance of multi-carrier communication systems, such as asymmetric digital subscriber lines (ADSL) [7] and power-line communication (PLC) [8], which face cumbersome impulsive noise scenarios [9]–[11].

II. SYSTEM MODEL

Let's consider a source x with a zero-mean Gaussian *pdf* $f_X(x) = G(x; \sigma_X^2) = (\sqrt{2\pi}\sigma_X)^{-1}e^{-x^2/2\sigma_X^2}$, impaired by a Class-A impulsive noise n , as expressed by

$$y = x + n, \quad (1)$$

and $\hat{x} = g(y)$ a possible estimator of x , as shown in Fig. 1. Specifically, the Class-A noise *pdf* is expressed by [1]

$$f_N(n) = \sum_{m=0}^{\infty} \beta_m G(n; \sigma_m^2) = \sum_{m=0}^{\infty} \frac{\beta_m}{\sqrt{2\pi}\sigma_m^2} e^{-\frac{n^2}{2\sigma_m^2}}, \quad (2)$$

which is clearly a Gaussian-mixture, where $\beta_m = e^{-A} A^m / m!$ is the Poisson-distributed probability that m noise sources simultaneously contribute to the impulsive event, and $A = E\{m\} = \sum_{m=0}^{\infty} m\beta_m$ is the corresponding expected value [2]. Moreover, $\sigma_N^2 = E\{n^2\} = \sigma_I^2 + \sigma_t^2$ is the noise power, where σ_I^2 is the impulsive power, $T = \sigma_t^2 / \sigma_I^2$ is the power-ratio with the AWGN, and $\sigma_m^2 = (m/A + T) / (1 + T) \sigma_N^2 = m\sigma_I^2 / A + \sigma_t^2$. Thus, the Class-A model is totally characterized by the canonical parameters A , T , and σ_N^2 . Specifically, low values of A identify rare and highly peaked impulsive noises and, conversely, high values of A makes the noise more similar to an AWGN [2].

III. OPTIMUM BAYESIAN ESTIMATOR (OBE)

Exploiting Bayes rules, the MMSE Bayesian estimator of x given the observation y , is expressed by [12]

$$\hat{x}_{\text{OBE}}(y) = E\{x|y\} = \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} x f_{Y|X}(y|x) f_X(x) dx, \quad (3)$$

where $f_{Y|X}(y|x)$ represents the conditional *pdf* of the observed data y for a given source x . Due to the fact that the impulsive noise n is independent of x , it is well known that $f_Y(y) = f_X(y) * f_N(y)$ [13], as expressed by

$$\begin{aligned} f_Y(y) &= \sum_{m=0}^{\infty} \beta_m G(y; \sigma_m^2) * G(y; \sigma_X^2) \\ &= \sum_{m=0}^{\infty} \frac{\beta_m}{\sqrt{2\pi}(\sigma_m^2 + \sigma_X^2)} e^{-\frac{y^2}{2(\sigma_m^2 + \sigma_X^2)}}, \end{aligned} \quad (4)$$

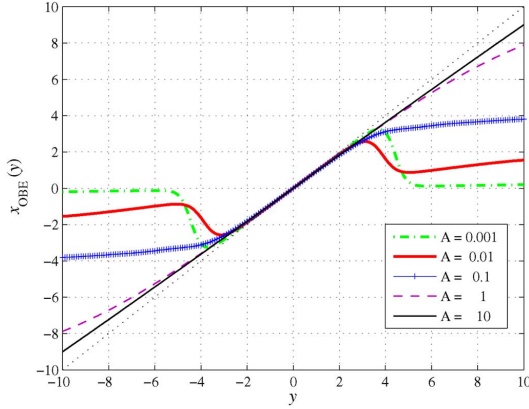


Fig. 2. OBE for $T = 1$, $\sigma_X^2 = 10$, $\sigma_N^2 = 1$ and several values of A .

where $*$ stands for the convolution integral, and it is exploited that the convolution of two Gaussian *pdfs* generates another Gaussian *pdf*, with sum of the variances [13]. Observing (1), it is also evident that $f_{Y|X}(y; x)$ is expressed by $f_{Y|X}(y; x) = f_N(y - x)$, which plugged in (3) leads to

$$\hat{x}_{\text{OBE}}(y) = \frac{1}{f_Y(y)} \sum_{m=0}^{\infty} \beta_m \int_{-\infty}^{\infty} x f_X(x) f_m(y - x) dx. \quad (5)$$

By using $S(f) = \mathcal{F}\{s(x)\}$ to indicate the Fourier transform (FT) of $s(x)$, it is reminded that $\mathcal{F}\{G(x; \sigma_X^2)\} = \tilde{G}(f; \sigma_{X,f}^2) = \sqrt{2\pi} \sigma_{X,f} G(f; \sigma_{X,f}^2)$, with $\sigma_{X,f}^2 = (4\pi\sigma_X^2)^{-1}$. Thus, using $p(x) = x f_X(x)$, and exploiting FT properties, the FT of the integral in (5) is expressed by

$$\begin{aligned} \mathcal{F}\{p(x) * f_m(x)\} &= \mathcal{F}\{xG(x; \sigma_X^2)\} \mathcal{F}\{G(x; \sigma_m^2)\} \\ &= \frac{j}{2\pi} \frac{d}{df} \left[\tilde{G}(f; \sigma_{X,f}^2) \right] \tilde{G}(f; \sigma_{m,f}^2) \\ &= -\frac{j}{2\pi\sigma_{X,f}^2} f \tilde{G}(f; \sigma_{X,f}^2) \tilde{G}(f; \sigma_{m,f}^2), \quad (6) \end{aligned}$$

where the last equality comes from $\frac{d}{df}[G(f; \sigma^2)] = -(f/\sigma^2)G(f/\sigma^2)$. Consequently, by inverse-FT duality property

$$\begin{aligned} p(x) * f_m(x) &= \frac{-1}{4\pi^2\sigma_{X,f}^2} \frac{d}{dx} \left[\mathcal{F}^{-1} \left\{ \tilde{G}(f; \sigma_{X,f}^2) \tilde{G}(f; \sigma_{m,f}^2) \right\} \right] \\ &= -\sigma_X^2 \frac{d}{dx} [G(x; \sigma_X^2) * G(x; \sigma_m^2)] \\ &= -\sigma_X^2 \frac{d}{dx} G(x; \sigma_X^2 + \sigma_m^2) \\ &= \frac{\sigma_X^2}{\sigma_X^2 + \sigma_m^2} x G(x; \sigma_X^2 + \sigma_m^2). \quad (7) \end{aligned}$$

Summarizing, (5) can be expressed by

$$\hat{x}_{\text{OBE}}(y) = \frac{\sum_{m=0}^{\infty} \frac{\sigma_X^2}{\sigma_X^2 + \sigma_m^2} \beta_m G(y; \sigma_X^2 + \sigma_m^2)}{\sum_{m=0}^{\infty} \beta_m G(y; \sigma_X^2 + \sigma_m^2)} y. \quad (8)$$

Equation (8) highlights how the OBE depends on the source average power σ_X^2 , and the noise canonical parameters A , T , and σ_N^2 , through β_m and σ_m^2 . The input-output characteristic of the OBE is plotted in Fig. 2 for several values of the parameter

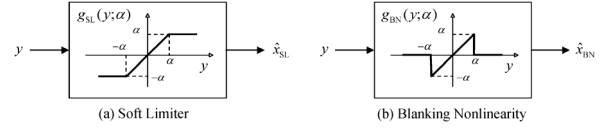


Fig. 3. Suboptimal estimators.

A , which controls the peakness of the impulsive noise [2]; it is evident that for high values of A , when $f_N(n)$ tends to a zero-mean Gaussian *pdf*, the OBE tends to the well known linear-MMSE estimator [12], expressed by

$$\hat{x}_{\text{OBE}}^{(lin)}(y) = \frac{E\{xy\}}{E\{y^2\}} y = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_N^2} y. \quad (9)$$

Conversely, for lower values of A , when the noise is characterized by rare and highly peaked impulses, the OBE shows a highly non-linear nature, by roughly limiting ($A = 0.1$), or blanking ($A = 0.001$), the observed values y that overpass certain thresholds. Noteworthy, the OBE in (8) is easily extended to any other Gaussian-mixture noise n .

IV. BAYESIAN SOFT LIMITER ESTIMATOR (SLE)

To protect device integrity, it is often requested to contrast impulsive receiver noise before A/D conversion. In this case, implementation of the OBE in (8) by analogic hardware is rather complicated, especially if the OBE should be adaptive to changes of the average powers σ_X^2 and σ_N^2 , and/or the noise peakness factors A , and T . Furthermore, also when digital implementations after A/D conversion is possible, the use of (8) may be prevented by either memory, or computational, or power-budget constraints. This is the case for instance of battery-powered sensors, deployed in huge number in the environment to monitor some physical parameter for a long time, and where battery life and very low-cost is an issue. Thus, this section investigates a simple suboptimum estimator, namely the SL shown in Fig. 3(a), which is widely employed to contrast impulsive noise [14], and adds robustness to the system by clipping the signal values exceeding a given threshold α . In this case, the only parameter to optimize in the Bayesian sense is the clipping threshold α , which obviously would depend on the noise parameters A , T , and σ_N^2 , as well as on the source power σ_X^2 . Meaningfulness of such an optimization, which leads to the SLE, is also suggested by the OBE shapes in Fig. 2, which for certain noise parameters (e.g., $A \in [0.1, 1]$) resemble the SL estimator of Fig. 3(a).

The output of the SL in Fig. 3(a) is expressed by a non-linear input-output characteristic $\hat{x}_{\text{SL}} = g_{\text{SL}}(y; \alpha)$. The SL estimation error $e_{\text{SL}}(x, n; \alpha) = x - g_{\text{SL}}(x + n; \alpha)$ depends on the selected threshold α , as well as on the statistical properties of the source x and the noise n . This is expressed by

$$e_{\text{SL}}(x, n; \alpha) = \begin{cases} x + \alpha, & x + n < -\alpha \\ -n, & |x + n| \leq \alpha \\ x - \alpha, & x + n > \alpha. \end{cases} \quad (10)$$

The SLE estimator is defined by selecting the threshold

$$\alpha_{\text{SL}}^{(mse)} = \arg \min_{\alpha \in \mathcal{R}^+} [J_{\text{SL}}(\alpha)] = \arg \min_{\alpha \in \mathcal{R}^+} [E\{e_{\text{SL}}^2(x, n; \alpha)\}] \quad (11)$$

that minimizes the MSE cost function $J_{\text{SL}}(\alpha)$. Thus, in order to find the optimal threshold $\alpha_{\text{SL}}^{(mse)}$ it is necessary to solve

$$\begin{aligned} \frac{\partial}{\partial \alpha} E \{ e_{\text{SL}}^2(x, n; \alpha) \} &= E \left\{ 2e_{\text{SL}}(x, n; \alpha) e_{\text{SL}}^{(1, \alpha)}(x, n; \alpha) \right\} \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e_{\text{SL}}(x, n; \alpha) e_{\text{SL}}^{(1, \alpha)}(x, n; \alpha) \\ &\quad \times f_X(x) f_N(n) dx dn = 0, \end{aligned} \quad (12)$$

where $e_{\text{SL}}^{(1, \alpha)}(x, n; \alpha)$ stands for the first partial derivative of $e_{\text{SL}}(x, n; \alpha)$ with respect to α . By substituting (2), (10), and its partial derivative in (12), after tedious derivations detailed in [15], it is proved that $\alpha_{\text{SL}}^{(mse)}$ is the solution of the following fixed-point (FP) equation

$$\alpha = F_{\text{SL}}^{(mse)}(\alpha) = \frac{2\sigma_X^2 \sum_{m=0}^{\infty} \frac{\beta_m}{\sqrt{2\pi(\sigma_X^2 + \sigma_m^2)}} e^{-\frac{\alpha^2}{2(\sigma_X^2 + \sigma_m^2)}}}{1 - \sum_{m=0}^{\infty} \beta_m \operatorname{erf}\left(\frac{\alpha}{\sqrt{2(\sigma_X^2 + \sigma_m^2)}}\right)}. \quad (13)$$

Due to lack of space, it is omitted the proof that a solution of the FP (13) always exists, as detailed in [15], where it is also proved that $J_{\text{SL}}(\alpha)$ in (11) is locally convex for $\alpha \in [0, 2.05\sigma_X]$, i.e., that $F_{\text{SL}}^{(mse)}(\alpha)$ is locally a contraction mapping [6]. Thus, any iterative solution of (11) starting from $\alpha_0 \in [0, 2.05\sigma_X]$ converges to the MSE minimum, as the succession $\alpha_{n+1} = F_{\text{SL}}^{(mse)}(\alpha_n)$ converges to the exact FP solution $\alpha_{\text{SL}}^{(mse)}$ [6]. Consequently, $\alpha_{\text{SL}}^{(mse)}$ can be numerically approximated by the following iterative algorithm

A1: Iterative algorithm for optimal SL threshold

1. set $\alpha_0 = F_{\text{SL}}^{(mse)}(0)$ and $n = 0$;
2. while $|F_{\text{SL}}^{(mse)}(\alpha_n) - \alpha_n| > \varepsilon$ and $n \leq n_{\text{Max}}$
3. $\alpha_{n+1} = F_{\text{SL}}^{(mse)}(\alpha_n)$;
4. $n = n + 1$;
5. end
6. set $\alpha_{\text{SL}}^{(mse)} = \alpha_n$.

In algorithm A1, ε represents the accuracy that is requested for the approximated FP solution to stop within a maximum number n_{Max} of iterations. Obviously, other iterative numerical approaches can be used to solve (13), such as the Newton—Rapson method [6] to find the root of $F_{\text{SL}}^{(mse)}(\alpha) - \alpha = 0$, or equivalently to solve $F_{\text{SL}}^{(mse)}(\alpha)/\alpha = 1$.

V. BAYESIAN BLANKING ESTIMATOR (BNE)

Fig. 2 suggests that for highly impulsive noise behaviors (e.g., $A \in [0.001, 0.01]$), the OBE shape resembles another quite used estimator, i.e., the BN shown in Fig. 3(b) and proposed in [16]. Similarly to the SLE, the estimation error $e_{\text{BN}}(x, n; \alpha) = x - g_{\text{BN}}(x + n; \alpha)$ is expressed by

$$e_{\text{BN}}(x, n; \alpha) = \begin{cases} -n, & |x + n| \leq \alpha \\ 0, & |x + n| > \alpha, \end{cases} \quad (14)$$

and, as detailed in [15], the optimum $\alpha_{\text{BN}}^{(mse)}$ can be obtained as the solution of the FP equation

$$\alpha = F_{\text{BN}}^{(mse)}(\alpha) = \frac{\sum_{m=0}^{\infty} \frac{A^m}{m!} \frac{2\sigma_m^2}{(\sigma_X^2 + \sigma_m^2)^{\frac{3}{2}}} \alpha e^{-\frac{\alpha^2}{2(\sigma_X^2 + \sigma_m^2)}}}{\sum_{m=0}^{\infty} \frac{A^m}{m!} \frac{1}{(\sigma_X^2 + \sigma_m^2)^{\frac{1}{2}}} e^{-\frac{\alpha^2}{2(\sigma_X^2 + \sigma_m^2)}}}. \quad (15)$$

Although the FP problem admits a unique solution, as detailed in [15], differently from $F_{\text{SL}}^{(mse)}(\alpha)$, $F_{\text{BN}}^{(mse)}(\alpha)$ is not a contraction mapping and, consequently, $F_{\text{BN}}^{(mse)}(\alpha)$ in (15) is not an attraction for the iterative algorithm A1. However, defining $G_{\text{BN}}(\alpha) = F_{\text{BN}}^{(mse)}(\alpha)/\alpha$, an iterative algorithm that converges to the FP is obtained from A1 by setting $\alpha_0 = G_{\text{BN}}(0)$, and by substituting the 3rd step with

$$3. \alpha_{n+1} = \alpha_n + \mu [1 - G_{\text{BN}}(\alpha_n)].$$

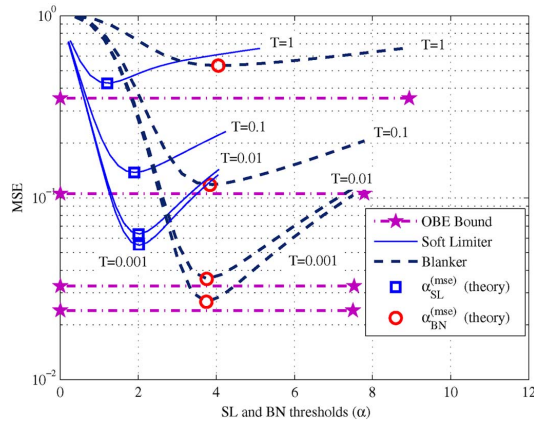
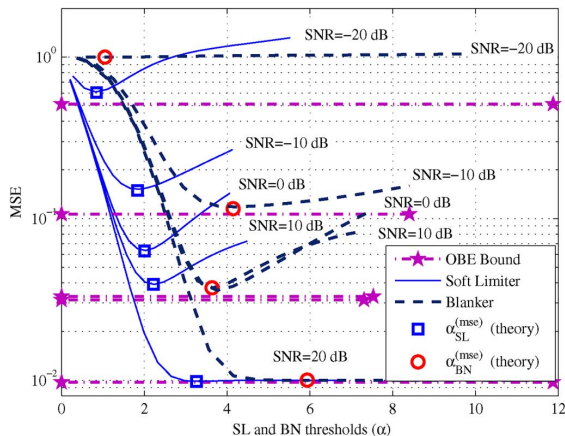
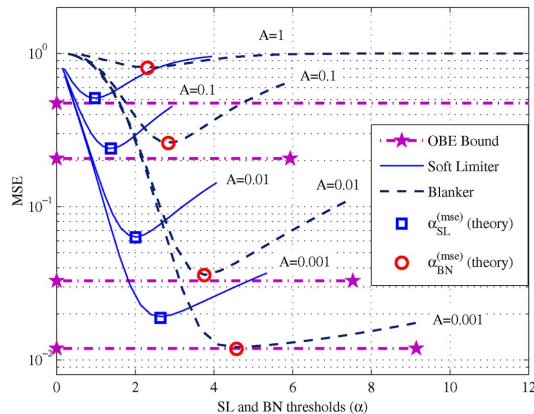
To quicken convergence, whose speed is controlled by $\mu \in]0, 1[$, the 3rd step now solves the equivalent problem $G_{\text{BN}}(\alpha) = 1$. Noteworthy, $G_{\text{BN}}(\alpha)$ increases monotonically with α [15]: thus, when $G_{\text{BN}}(0) > 1$ the (trivial) optimal BNE threshold is $\alpha_{\text{BN}}^{(mse)} = 0$.

VI. COMPUTER SIMULATIONS

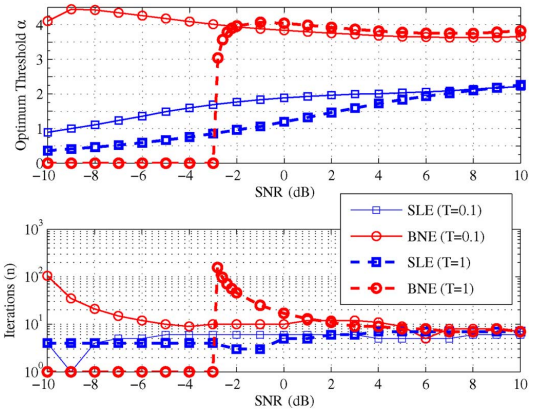
The Middleton's Class-A noise has been generated by the toolbox [17]. The optimal thresholds for the MMSE SL and BN are obtained by the algorithm A1 using $\varepsilon = 0.01$, $\mu = 0.99$ for the BN, and without loss of generality $\sigma_X^2 = 1$. Fig. 4–Fig. 6 show the sensitiveness of the MSE performance with respect to the thresholds values, for both the SL and the BN, and let to appreciate their performance penalty with respect to the OBE. The MSE plots in Fig. 4–Fig. 6 are obtained by generating 10^8 observed samples y in (1), in order to guarantee that the MSE sample-mean converges to the theoretical MSE (available in [15]) also when the impulsive noise is characterized by very rare events (i.e., by lower values of A). The squares and circles in Figs. 4–6 are the MSE values obtained for the thresholds computed by algorithm A1. As theoretically predicted, computer simulations confirm that these thresholds are optimal in a MMSE sense. Interestingly, at least one among the optimal SL and BN is characterized by negligible (minimum) MSE penalty with respect to the OBE, and could safely be used as a valuable low-complexity alternative. This fact was somehow suggested by the OBE shapes in Fig. 2, which highly resemble either a SL or a BN, for several values of the canonical Class-A parameters. Finally, Fig. 7 shows the optimal SL and BN thresholds, obtained when $A = 0.01$, together with the associated number of iterations requested by A1 to converge. Generally, convergence is really fast (especially for the SLE), although for some specific values of SNR, A , and T , the BN may request some longer time.

VII. CONCLUSIONS AND FUTURE WORK

The MMSE Bayesian estimator for a Gaussian source impaired by impulsive Middleton's Class-A interference has been derived. Furthermore, two popular and sub-optimal estimators, namely the soft-limiter and the blanker, have been optimized in the MMSE sense. The proposed Bayesian estimators can be easily adapted to any Gaussian source impaired by any Gaussian-mixture noise.


 Fig. 4. MSE curves for $A = 0.01$ and $\text{SNR} = 0$ dB.

 Fig. 5. MSE curves for $A = 0.01$ and $T = 0.01$.

 Fig. 6. MSE curves for $T = 0.001$ and $\text{SNR} = 0$ dB.

Other criteria, rather than the MMSE, could be used to set the optimal thresholds of the BN and the SL, as for instance the maximum-SNR (MSNR) criterion, proposed in [16] and [14]. Note that, while MMSE and MSNR are equivalent in pure AWGN scenarios [18], this is not the case when the noise is a Gaussian-mixture, which leads to a non-linear MMSE estimator (see also [15] and [19]). Whether it is better the MSNR or the MMSE criterion, depends on the specific application and design constraints: this investigation is left for future works where, in the light of [19], it will be also possible to establish theoretical expressions and relationships between the MSNR and MMSE estimators.


 Fig. 7. Optimal SL and BN thresholds ($A = 0.01$, $\sigma_X^2 = 1$).

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